

# Algebraic logic and logically-geometric types in varieties of algebras

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## Abstract

The main objective of this paper is to show that the notion of type which was developed within the frames of logic and model theory has deep ties with geometric properties of algebras. These ties go back and forth from universal algebraic geometry to the model theory through the machinery of algebraic logic. We show that types appear naturally as logical kernels in the Galois correspondence between filters in the Halmos algebra of first order formulas with equalities and elementary sets in the corresponding affine space.

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# 1 Introduction

The main objective of the paper is to show that the notion of type which was developed within the frames of logic and model theory has deep ties with geometric properties of algebras. These ties go back and forth from universal algebraic geometry to model theory through the machinery of algebraic logic.

More precisely, we shall show that types appear naturally as logical kernels in the Galois correspondence between filters in the Halmos algebra of first order formulas with equalities and elementary sets in the corresponding affine space. Note that in our terminology the term "elementary set" has the meaning of "definable set" in the standard model theoretic terminology. This Galois correspondence generalizes classical Galois correspondence between ideals in the polynomial algebra and algebraic sets in the affine space. The sketch of the ideas of universal algebraic geometry can be found in [31], [33], [34], [36], [37], [3], [29], [22], [7] [8], [9], [5], [6], [20], [21], [40], [41], etc. As for standard definitions of model theory, we refer to monographs [28], [39], [4], [17], etc. For the exposition of concepts and results of algebraic logic see [10] – [14], [15], [18], [19], [2], [1], etc.

Methodologically, in the paper we give a sketch of some ideas which provide interactions of algebraic logic with geometry, model theory and algebra. We believe that a development of the described approach can make benefits to each of these areas. We shall stress that the paper does not contain a bunch of new results. Its main duty is to specialize new problems and to underline common points of algebra, logic and geometry through the notion of the type.

The paper is organized as follows. Section 2 is devoted to structures of algebraic logic. We define here various kinds of Halmos algebras, consider the value homomorphism and provide the reader with the main examples of algebras under consideration. Section 3 deals with basic approaches of universal algebraic geometry. We define the general Galois correspondence which plays the important role in all considerations. The description of this correspondence starts from the classical case and extends to the case of multi-sorted logical geometry over an arbitrary variety of algebras. In Section 4 we recall the model theoretic notion of a type. In Section 5 we concentrate attention on types from the positions of one-sorted algebraic logic. Section 6 deals with the ideas of universal logical geometry which give rise to  $LG$ -types and their geometric description. We finish the paper with the list of problems appearing in the context of previous considerations.

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# 2 Structures of algebraic logic

We consider algebra and logic with respect to a given variety of algebras  $\Theta$ . This point of view (cf. [35]) implies some differences with the original notions introduced by P. Halmos ([10] – [14], see [24] for non-homogenous polyadic algebras). For the sake of convenience, in this section we provide the reader with

all necessary definitions. It will be emphasized that the transition from pure logic to logic in  $\Theta$  is caused by many reasons, and we would like to distinguish the needs of universal algebraic geometry among them.

Denote by  $\Omega$  the signature of operations in algebras from  $\Theta$ . Let  $W(X)$  denote the free in  $\Theta$  algebra over a non-empty set of variables  $X$ . In the meantime we assume that each  $X$  is a subset of some infinite set of variables  $X^0$ .

We shall recall the well-known definitions of the existential and universal quantifiers which are considered as new operations on Boolean algebras (see [10]).

Let  $B$  be a Boolean algebra. The mapping  $\exists : B \rightarrow B$  is called an *existential* quantifier if

1.  $\exists(0) = 0$ ,
2.  $a \leq \exists(a)$ ,
3.  $\exists(a \wedge \exists b) = \exists a \wedge \exists b$ .

The *universal* quantifier  $\forall : B \rightarrow B$  is defined dually:

1.  $\forall(1) = 1$ ,
2.  $a \geq \forall(a)$ ,
3.  $\forall(a \vee \forall b) = \forall a \vee \forall b$ .

Here the numerals 0 and 1 are zero and unit of the Boolean algebra  $B$  and  $a, b$  are arbitrary elements of  $B$ . Symbol  $=$  means coincidence of elements in Boolean algebra, i.e.,  $a \leq b$  and  $b \leq a$  is written as  $a = b$ ,  $a, b \in B$ . The quantifiers  $\exists$  and  $\forall$  are coordinated by:  $\neg(\exists a) = \forall(\neg a)$ , i.e.,  $(\forall a) = \neg(\exists(\neg a))$ .

A pair  $(B, \exists)$ , where  $B$  is a Boolean algebra and  $\exists$  is the existential quantifier, is a *monadic* algebra (see [10]).

**Definition 2.1** A Boolean algebra  $B$  is a quantifier  $X$ -algebra if a quantifier  $\exists x : B \rightarrow B$  is defined for every variable  $x \in X$ , and

$$\exists x \exists y = \exists y \exists x,$$

for every  $x, y \in X$ .

**Remark 2.2** See also the definition of diagonal-free cylindric algebras of Tarski e.a. [15].

**Remark 2.3** According to [10], [35] a Boolean algebra  $B$  is a quantifier  $X$ -algebra if a quantifier  $\exists(Y) : B \rightarrow B$  is defined for every subset  $Y \subset X$ , and

1.  $\exists(\emptyset) = I_B$ , the identity function on  $B$ ,
2.  $\exists(X_1 \cup X_2) = \exists(X_1)\exists(X_2)$ , where  $X_1, X_2$  are subsets in  $X$ .

If we restrict ourselves with finite nontrivial subsets of  $X$ , then these two definitions coincide, because condition 2) implies commutativity of quantifiers, and, conversely, one can define  $\exists(Y) = \exists y_1 \cdots \exists y_k$ , where  $Y = \{y_1, \dots, y_k\}$ .

We shall consider also *quantifier  $W(X)$ -algebras  $B$  with equalities*. An equality in a quantifier  $W(X)$ -algebra is symmetric, reflexive and transitive (see Definition 2.4) predicate  $\equiv: W(X) \times W(X) \rightarrow B$  which takes a pair  $w, w' \in W(X)$  to the constant in  $B$  denoted by  $w \equiv w'$ , subject to condition:

1)  $(w_1 \equiv w'_1 \wedge \dots \wedge w_n \equiv w'_n) \leq (w_1 \dots w_n \omega \equiv w'_1 \dots w'_n \omega)$  where  $\omega$  is an  $n$ -ary operation in  $\Omega$ .

We can speak about quantifier  $W(X)$ -algebras, assuming that the free in  $\Theta$  algebra  $W(X)$  uniquely corresponds to each set  $X$ . Suppose that the logical signature is extended by symbols of nullary operations  $w \equiv w'$ , where  $w, w' \in W(X)$ . Then

**Definition 2.4** *We call a Boolean algebra  $B$  a quantifier  $W(X)$ -algebra with equalities (or an extended Boolean algebra over the free in  $\Theta$  algebra  $W(X)$ ), if*

1. *There are defined quantifiers  $\exists x$  for all  $x \in X$  in  $B$  with  $\exists x \exists y = \exists y \exists x$  for all  $x, y \in X$ .*

2. *To every pair  $w, w' \in W(X)$  it corresponds a constant (called an equality) in  $B$ , denoted by  $w \equiv w'$ . Here,*

2.1.  $w_1 \equiv w'_1 \leq w'_1 \equiv w_1$ .

2.2.  $w \equiv w$  is the unit of the algebra  $B$ .

2.3.  $w_1 \equiv w_2 \wedge w_2 \equiv w_3 \leq w_1 \equiv w_3$ .

2.4. *For every  $n$ -ary operation  $\omega \in \Omega$ , where  $\Omega$  is a signature of the variety  $\Theta$ , we have*

$$w_1 \equiv w'_1 \wedge \dots \wedge w_n \equiv w'_n \leq w_1 \dots w_n \omega \equiv w'_1 \dots w'_n \omega.$$

**Remark 2.5** *Under homomorphisms of extended Boolean algebras each constant  $w \equiv w'$  goes to another constant of the same kind. Endomorphisms of Boolean algebras leave constants  $w \equiv w'$  unchanged.*

**Remark 2.6** *Condition 2.4 means that for every homomorphism  $\mu: W(X) \rightarrow H$ , where  $H \in \Theta$ , there is a coordination of  $\mu$  with all operations from  $\Omega$ . In other words equalities respect all operations on  $W(X)$ .*

**Definition 2.7** *An algebra  $\mathfrak{L} = \mathfrak{L}(X)$  is a Halmos algebra (one-sorted Halmos algebra) over  $W(X)$ ,  $X$  is infinite if:*

1.  $\mathfrak{L}$  is an extended Boolean algebra.
2. *The action of the semigroup  $\text{End}(W(X))$  is defined on  $\mathfrak{L}$ , so that for each  $s \in \text{End}(W(X))$  there is the map  $s_*: \mathfrak{L} \rightarrow \mathfrak{L}$  which preserves the Boolean structure of  $\mathfrak{L}$ .*
3. *The identities controlling the interaction of  $s_*$  with quantifiers are as follows:*

3.1.  $s_{1*} \exists x a = s_{2*} \exists x a$ ,  $a \in \mathfrak{L}$ , if  $s_1(y) = s_2(y)$  for every  $y \in X$ ,  $y \neq x$ .

3.2.  $s_* \exists x a = \exists(s(x))(s_* a)$ ,  $a \in \mathfrak{L}$ , if  $s(x) = y$  and  $y$  is a variable which does not belong to the support of  $s(x')$ , for every  $x' \in X$ , and  $x' \neq x$ .

*This condition means that  $y$  does not participate in the shortest expression of the element  $s(x') \in W(X)$  through the elements of  $X$ .*

4. The identities controlling the interaction of  $s_*$  with equalities are as follows:

$$4.1 \quad s_*(w \equiv w') = (s(w) \equiv s(w')).$$

$$4.2 \quad (s_w^x)_* a \wedge (w \equiv w') \leq (s_{w'}^x)_* a, \text{ where } a \in \mathfrak{L}, \text{ and } s_w^x \in \text{End}(W(X)) \text{ is defined by } s_w^x(x) = w, \text{ and } s_w^x(x') = x', \text{ for } x' \neq x.$$

**Remark 2.8** The set  $X$  in the definition 2.7 must be infinite because otherwise  $\text{End}(W(X))$  does not act on  $B$  (see [35], Chapter 8, Section 2 for the details) in the case of free Halmos algebras. In general this condition is superfluous since we require the action of the semigroup  $\text{End}(W(X))$  on the algebra  $\mathfrak{L}$ .

For the definition of support see [35], Chapter 9, Section 1.

**Remark 2.9** Definition 2.7 introduces algebras which are very close to polyadic algebras of Halmos (see [10]) defined over a set of variables  $X$ . The main difference between these classes comes from the desire to specialize an algebraization of first order logic to an arbitrary variety of algebras  $\Theta$ . This means that instead of action of the semigroup of transformations  $\text{End}(X)$  of the set  $X$ , we consider the action of the bigger semigroup  $\text{End}(W(X))$  as the semigroup of Boolean endomorphisms. We also consider equalities of the type  $w \equiv w'$  instead of the ones  $x \equiv y$  for polyadic algebras.

**Remark 2.10** Axioms 3.1 and 3.2 which look messy, are grounded on major examples of Halmos algebras. In particular, we will see that Halmos algebras of the kind  $\text{Hal}_\Theta(H)$  (see Example 2.12) satisfy these identities. Since these algebras generate the whole variety of Halmos algebras, every Halmos algebra should satisfy these identities. If instead of  $\text{Hal}_\Theta(H)$  we consider the Halmos algebra of formulas  $\widehat{\Phi}$  (see below), then the identity 3.1. corresponds to the well-known fact that it is possible to replace a quantified variable in a formula by another one. The identity 3.2. has a similar explanation (see [10]).

**Remark 2.11** In [10], [35] an equality in Halmos algebras is defined as a reflexive binary predicate which satisfies conditions 4.1. and 4.2. Then, it can be checked [35], that this predicate is automatically symmetric and transitive.

**Example 2.12** We give an example of Halmos algebra which plays a crucial role in further considerations.

Let  $X$  be any set (finite or infinite),  $H$  an algebra in  $\Theta$ . Consider the set  $\text{Hom}(W(X), H)$  of all homomorphisms from  $W(X)$  to  $H$ . Let  $\text{Bool}(W(X), H)$  be the Boolean algebra of all subsets  $A$  in  $\text{Hom}(W(X), H)$ . Our aim is to make it an extended Boolean algebra.

Define, first, quantifiers  $\exists x$ ,  $x \in X$  on  $\text{Bool}(W(X), H)$ . We set  $\mu \in \exists x A$  if and only if there exists  $\nu \in A$  such that  $\mu(y) = \nu(y)$  for every  $y \in X$ ,  $y \neq x$ . It can be checked that  $\exists x$  defined in such a way is, indeed, an existential quantifier.

Let us consider equalities of the form  $w \equiv w'$ , where  $w, w' \in W(X)$ . Define the corresponding elements of the algebra  $\text{Bool}(W(X), H)$  as follows

$$\text{Val}_H^X(w \equiv w') = \{\mu \mid \mu(w) = \mu(w')\}.$$

The set  $\text{Val}_H^X(w \equiv w')$  is considered as an equality in the algebra  $\text{Bool}(W(X), H)$ .

Thus, the algebra  $Bool(W(X), H)$  is equipped with the structure of an extended Boolean algebra (we omit verification of the necessary axioms).

Let  $X$  now be an infinite set. Define the action of the semigroup  $End(W(X))$  in  $Bool(W(X), H)$ . Every homomorphism  $s \in End(W(X))$  gives rise to a Boolean homomorphism

$$s_* : Bool(W(X), H) \rightarrow Bool(W(X), H),$$

defined by the rule: for each  $A \subset Hom(W(X), H)$  the point  $\mu$  belongs to  $s_*A$  if  $\mu s \in A$ .

The signature of a Halmos algebra for  $Bool(W(X), H)$  is now completed, one can check that all axioms are satisfied and thus,  $Bool(W(X), H)$  is a Halmos algebra. Denote it by  $Hal_{\Theta}^X(H)$ .

Our next aim is to define multi-sorted Halmos algebras. There are many reasons to do that. Some of them are related to potential applications of algebraic logic in computer science, but some have purely algebraic nature. For instance, we need multi-sorted variant of Halmos algebras in order to work with finite dimensional affine spaces and to construct geometry related to first order calculus in arbitrary  $\Theta$ .

Every multi-sorted algebra  $D$  can be written as  $D = (D_i, i \in \Gamma)$ , where  $\Gamma$  is a set of sorts, which can be infinite, and  $D_i$  is a domain of the sort  $i$ . We can regard domains  $D_i$  as algebras from some variety (for definitions see [23], [26]).

Every operation  $\omega$  in  $D$  has a specific type  $\tau = \tau(\omega)$ . This notion generalizes the notion of the arity of an operation. In the multi-sorted case an operation  $\omega$  of the type  $\tau = (i_1, \dots, i_n; j)$  operates as a mapping  $\omega : D_{i_1} \times \dots \times D_{i_n} \rightarrow D_j$ . Homomorphisms of multi-sorted algebras act component-wise and have the form  $\mu = (\mu_i, i \in \Gamma) : D \rightarrow D'$ , where  $\mu_i : D_i \rightarrow D'_i$  are homomorphisms of algebras and, besides that, every  $\mu$  is naturally correlated with the operations  $\omega$ .

Subalgebras, quotient algebras, and cartesian products of multi-sorted algebras are defined in the usual way. Hence, one can define varieties of multi-sorted algebras. In every such a variety there exist free algebras over multi-sorted sets, determined by multi-sorted identities.

It is worth noting that categories and multi-sorted algebras are tightly connected [16], [25]. So, define, first, Halmos categories. Let  $\Theta^0$  be the category of free algebras of the variety  $\Theta$ .

**Definition 2.13** *A category  $\Upsilon$  is a Halmos category if:*

1. *Every its object has the form  $\Upsilon(X)$ , where  $\Upsilon(X)$  is an extended Boolean algebra in  $\Theta$  over  $W(X)$ .*
2. *Morphisms are of the form  $s_* : \Upsilon(X) \rightarrow \Upsilon(Y)$ , where every  $s : W(X) \rightarrow W(Y)$  is a homomorphism in  $\Theta^0$ ,  $s_*$  is the homomorphism of Boolean algebras and the correspondence:  $W(X) \rightarrow \Upsilon(X)$  and  $s \rightarrow s_*$  determines a covariant functor  $\Theta^0 \rightarrow \Upsilon$ .*
3. *The identities controlling the interaction of morphisms with quantifiers and equalities repeat the ones from Definition 2.7, where the endomorphisms  $s$  from  $End(W(X))$  are replaced by homomorphisms  $s : W(X) \rightarrow W(Y)$ .*

Now we are able to define multi-sorted Halmos algebras associated with Halmos categories. Consider an arbitrary  $W(X)$  in  $\Theta$  and take the signature  $L_X = \{\vee, \wedge, \neg, \exists x, x \in X, M_X\}$ . Here  $M_X$  is the set of all equalities  $w \equiv w'$ ,  $w, w' \in W(X)$  over the algebra  $W(X)$ . We treat equalities from  $M_X$  as nullary operations. We add all  $s = s^{XY} : W(X) \rightarrow W(Y)$  to all  $L_X$ , where  $X, Y \in \Gamma$ , treating them as symbols of unary operations (under unary we mean that these operations of the type  $(X, Y)$  use just one argument). Denote the new signature by  $L_\Theta$ . So,

$$L_\Theta = \{\vee, \wedge, \neg, \exists x, x \in X, M_X, X \in \Gamma, s = s^{XY}\}$$

The signature  $L_\Theta$  is a multi-sorted signature and consists of all one-sorted signatures  $L_X$ , where  $X$  runs  $\Gamma$ , and of all  $s$ .

For the aims of logical geometry we assume that  $\Gamma$  is the set of all finite subsets of the infinite set  $X^0$ .

**Remark 2.14** *This condition on the domains  $\Gamma$  is not necessary for the definition of Halmos algebras and made exclusively for geometric needs. Halmos algebras can be defined for various choice of domains. For example, the one-sorted Halmos algebra from Definition 2.7 corresponds to the signature  $L_X = \{\vee, \wedge, \neg, \exists x, x \in X, M_X, s\}$ , where  $s : W(X) \rightarrow W(X)$ , and  $X = X^0$  is an infinite set.*

Consider further algebras  $\Upsilon = (\Upsilon_X, X \in \Gamma)$ . Every  $\Upsilon_X$  is an algebra in the signature  $L_X$  and a unary operation (mapping)  $s_* : \Upsilon_X \rightarrow \Upsilon_Y$  corresponds to every  $s : W(X) \rightarrow W(Y)$ .

**Definition 2.15** *We call an algebra  $\Upsilon = (\Upsilon_X, X \in \Gamma)$  in the signature  $L_\Theta$  a Halmos algebra, if*

1. *Every  $\Upsilon_X$  is an extended Boolean algebra in the signature  $L_X$ .*
2. *Every mapping  $s_* : \Upsilon_X \rightarrow \Upsilon_Y$  is a homomorphism of Boolean algebras.*
3. *The identities, controlling interaction of operations  $s_*$  with quantifiers and equalities are the same as in the definition of Halmos categories.*
4. *Let  $s : W(X) \rightarrow W(Y)$ ,  $s' : W(Y) \rightarrow W(Z)$ , and let  $u \in \Upsilon_X$ . Then  $s'_*(s_*(u)) = (s's)_*(u)$ .*

It is clear that each Halmos category  $\Upsilon$  can be viewed as a Halmos algebra and vice versa.

**Remark 2.16** *The choice of  $\Theta$  gives rise to some conditions all  $s_*$  have to satisfy.*

Now we shall construct two major examples of multi-sorted Halmos algebras. The first one mimics the construction of one-sorted Halmos algebra from Example 2.12.

1. Our aim is to define the Halmos category  $Hal_\Theta(H)$ . Assume that we have a class of sets  $X_i$ ,  $X_i \in \Gamma$ . Objects of this category are extended Boolean algebras  $Bool(W(X_i), H)$  from Example 2.12, for various  $X_i \in \Gamma$ . Morphisms

$$s_* : Bool(W(X_i), H) \rightarrow Bool(W(X_j), H),$$

are defined as follows:

$$\mu \in s_* A \Leftrightarrow \tilde{s}(\mu) = \mu s \in A,$$

where  $\mu : W(X_j) \rightarrow H$ ,  $A \subset \text{Hom}(W(X_i), H)$ ,  $s : W(X_i) \rightarrow W(X_j)$ , and  $\tilde{s} : \text{Hom}(W(X_j), H) \rightarrow \text{Hom}(W(X_i), H)$ . Here  $\tilde{s}$  is viewed as a morphism of the category of affine spaces. In other words,  $s_*A = (\tilde{s})^{-1}A$ . A morphism  $s_*$  is automatically a homomorphism of Boolean algebras. The maps  $s_*$  are correlated with quantifiers and equalities, see [31] for details. Moreover, there is a covariant functor:  $\Theta^0 \rightarrow \text{Hal}_\Theta(H)$ . Hence,  $\text{Hal}_\Theta(H)$  is a Halmos category.

The category  $\text{Hal}_\Theta(H)$  gives rise to a multi-sorted ( $\Gamma$ -sorted) Halmos algebra, denoted by

$$\text{Hal}_\Theta(H) = (\text{Bool}(W(X_i), H), X_i \in \Gamma).$$

Each component here is the extended Boolean algebra. The operations in  $\text{Hal}_\Theta(H)$  are presented by the operations in each component  $\text{Bool}(W(X_i), H)$  and unary operations corresponding to morphisms

$$s_* : \text{Bool}(W(X_i), H) \rightarrow \text{Bool}(W(X_j), H).$$

**2.** Another important example of multi-sorted Halmos algebra is presented by algebra  $\tilde{\Phi} = (\Phi(X), X \in \Gamma)$  of first order formulas with equalities. It turns out that geometrical aims forces to consider multi-sorted variant of algebraization of first order calculus and consider multi-sorted, in a special sense, formulas. We shall return to this discussion at the end of the section.

Consider once again the signature  $L_\Theta = \{\vee, \wedge, \neg, \exists x, x \in X, M_X, X \in \Gamma, s = s^{XY}\}$ , where  $M_X$  is the set of all equalities  $w \equiv w'$ ,  $w, w' \in W(X)$  over  $W(X)$  treated as nullary operations, and  $s = s^{XY} : W(X) \rightarrow W(Y)$  are symbols of unary operations.

First, we construct the algebra  $\tilde{\Phi}$  in an explicit way. Denote by  $M = (M_X, X \in \Gamma)$  the multi-sorted set of equalities with the components  $M_X$ .

Each equality  $w \equiv w'$  is a formula of the length zero, and of the sort  $X$  if  $w \equiv w' \in M_X$ . Let  $u$  be a formula of the length  $n$  and the sort  $X$ . Then the formulas  $\neg u$  and  $\exists x u$  are the formulas of the same sort  $X$  and the length  $(n+1)$ . Further, for the given  $s : W(X) \rightarrow W(Y)$  we have the formula  $s_*u$  with the length  $(n+1)$  and the sort  $Y$ . Let now  $u_1$  and  $u_2$  be formulas of the same sort  $X$  and the length  $n_1$  and  $n_2$  accordingly. Then the formulas  $u_1 \vee u_2$  and  $u_1 \wedge u_2$  have the length  $(n_1 + n_2 + 1)$  and the sort  $X$ . In such a way, by induction, we define lengths and sorts of arbitrary formulas.

Let  $\mathfrak{L}_X^0$  be the set of all formulas of the sort  $X$ . Each  $\mathfrak{L}_X^0$  is an algebra in the signature  $L_X$  and

$$\mathfrak{L}^0 = (\mathfrak{L}_X^0, X \in \Gamma)$$

is an algebra in the signature  $L_\Theta$ . By construction, algebra  $\mathfrak{L}^0$  is the absolutely free algebra of formulas over equalities (i.e. over nullary operations) concerned with the variety of algebras  $\Theta$ .

Denote by  $\tilde{\pi}$  the congruence in  $\mathfrak{L}^0$  generated by the identities of Halmos algebras from Definition 2.15 (see also their list in Definition 2.7) and define the Halmos algebra of formulas as

$$\tilde{\Phi} = \mathfrak{L}^0 / \tilde{\pi};$$

It can be written as  $\tilde{\Phi} = (\Phi(X), X \in \Gamma)$ , where

$$\Phi(X) = \mathfrak{L}_X^0 / \tilde{\pi}_X,$$



where each  $\Phi(X)$  is an extended Boolean algebra of the sort  $X$  in the signature  $L_X$ . The algebra  $\tilde{\Phi}$  is, obviously, the free algebra in the variety of all multi-sorted Halmos algebras associated with the variety of algebras  $\Theta$ , with the set of free generators  $M = (M_X, X \in \Gamma)$ . Denote this variety by  $Hal_\Theta$ .

**Remark 2.17** *One can show [32], that if we factor out component-wisely the algebra  $\mathfrak{L}^0$  by the many-sorted Lindenbaum-Tarski congruence, then we get the same algebra  $\tilde{\Phi}$ . This observation provides a bridge between syntactical and semantical description of the free multi-sorted Halmos algebra.*

**Remark 2.18** *To the contrary of the one-sorted case, the described construction does not give much practical information about multi-sorted formulas. Indeed, suppose we consider a one-sorted algebra  $\Phi(X)$ . Let us pick up an arbitrary element  $u$  from  $\Phi(X)$ . We can consider this element as a mirror in the one-sorted Halmos algebra  $\Phi(X)$  of a first order formula constructed on the base of the equality predicate. Looking at the element we can deduce the structure of the corresponding formula.*

*The existence of operations  $s : W(X) \rightarrow W(Y)$  breaks this intuition in many-sorted case. If an element  $u$  has the sort  $X$  and thus belong to  $\Phi(X)$ , then we cannot represent explicitly the element  $s_*u$  from  $\Phi(X)$  in terms of equalities, connectives, and quantifiers in  $\Phi(X)$ . This means that we cannot trace the structure of an arbitrary element from  $\Phi(X)$ .*

Fortunately, there exists a way out from the difficulty described in Remark 2.18. If we were to know what the algebras which constitute the variety  $Hal_\Theta$  are, then we could calculate the image of any element from  $\Phi(X)$  in algebras from  $Hal_\Theta$ . The following theorem yields that this is the case in our situation.

**Theorem 2.19 ([31])** *The variety  $Hal_\Theta$  of multi-sorted Halmos algebras is generated by all algebras  $Hal_\Theta(H)$ , where  $H \in \Theta$ .*

Theorem 2.19, in fact, gives us another definition for the algebra  $\tilde{\Phi}$ , which can be considered as a free algebra in the variety generated by algebras  $Hal_\Theta(H)$ . This allows us to study properties of  $\tilde{\Phi}$  using the very concrete algebra

$$Hal_\Theta(H) = (Bool(W(X), H), X \in \Gamma)$$

as a model. Recall that we have defined the image of equalities from  $M_X$  in  $Bool(W(X), H)$  by:

$$Val_H^X(w \equiv w') = \{\mu \mid \mu(w) = \mu(w')\}.$$

This means that there is the map

$$Val_H : M \rightarrow Hal_\Theta(H).$$

Since equalities  $M = (M_X, X \in \Gamma)$  freely generate the free multi-sorted Halmos algebra  $\tilde{\Phi}$ , the map  $Val_H$  can be extended from generators to the homomorphism of multi-sorted Halmos algebras

$$Val_H : \tilde{\Phi} \rightarrow Hal_\Theta(H).$$

Since  $\tilde{\Phi} = (\Phi(X), X \in \Gamma)$ , where each component  $\Phi(X)$  is an extended Boolean algebra, the homomorphism  $Val_H$  induces homomorphisms

$$Val_H^X : \Phi(X) \rightarrow Bool(W(X), H),$$

of the one-sorted extended Boolean algebras. This allows us to calculate the value of each element from  $\Phi(X)$  in  $Bool(W(X), H)$ . Note that the values of elements of the form  $s_*u$  are calculated as follows. Take  $s : W(X) \rightarrow W(Y)$  and consider the formula  $s_*u$ , where  $u \in \Phi(X)$ . By definition,  $s_*u$  belongs to  $\Phi(Y)$ . Since  $Val_H$  is a homomorphism, then

$$Val_H^Y(s_*u) = s_*(Val_H^X u).$$

In the next sections we shall put all this staff in the context of affine spaces in arbitrary varieties. Replacing usual equations by logical formulas we arrive at the field of logical geometry which is much more complicated than the ordinary equational geometry.

### 3 Structures of universal algebraic geometry

Let us begin with the very classical setting (cf. [42]). Let  $K$  be a field and  $T = \{f_1, \dots, f_m\}$  be a set polynomials in the polynomial algebra  $K[X] = K[x_1, \dots, x_n]$ . Consider the affine space  $K^n$  with points  $\bar{a} = (a_1, \dots, a_n)$ ,  $a_i \in K$  and define the Galois correspondence between ideals  $T$  in  $K[X]$  and algebraic sets  $A$  in  $K^n$ :

$$T'_K = A = \{\bar{a} \mid f_i(\bar{a}) = 0, \text{ for all } f_i \in T\},$$

$$A'_K = T = \{f_i \in K[X] \mid f_i(\bar{a}) = 0, \text{ for all } \bar{a} \in A\},$$

In this correspondence geometric objects: curves, surfaces, general algebraic sets appear as zero loci of polynomials in the algebra  $K[X]$ .

In order to generalize this situation to arbitrary varieties of algebras, consider the variety  $Com-K$  of commutative, associative algebras with unit over the field  $K$ . Then the algebra  $K[X]$  is the free algebra in this variety and polynomials  $f_i$  are just elements of free algebra. Consider the field  $K$  and its extensions as algebras in this variety. Consider elements  $\bar{a} = (a_1, \dots, a_n)$  of the affine space  $K^n$  as functions  $\bar{a} : K[X] \rightarrow K$  defined by  $\bar{a}(x_i) = a_i$ ,  $i = 1, \dots, n$ . Using this vocabulary we can define the Galois correspondence and geometric objects not in  $Com - P$  but in arbitrary  $\Theta$ .

Let  $\Theta$  be an arbitrary variety and  $H$  be an algebra in  $\Theta$ . This algebra takes the role of the field  $K$ , hence the affine space has to be of the form  $H^n$ . Let  $W(X)$  be the free algebra over  $X$ ,  $X = \{x_1, \dots, x_n\}$ . This is the place where equations are situated and thus it plays the role of  $K[x_1, \dots, x_n]$ . The natural bijection  $\alpha : Hom(W(X), H) \rightarrow H^n$  allows us to consider the set of homomorphisms  $Hom(W(X), H)$  as the affine space and its elements as the points of the affine space. Let the point  $\mu \in Hom(W(X), H)$  be induced by a map  $\mu : X \rightarrow H$ . Then it corresponds the point  $\bar{a} = (a_1, \dots, a_n)$  in  $H^n$ , where  $a_i = \mu(x_i)$ . This correspondence gives rise to kernels of points  $\mu$  of the affine space. We define the kernel  $Ker(\mu)$  of the point  $\mu$  as the kernel of the homomorphism  $\mu : W(X) \rightarrow H$ .

Let  $T$  be a system of equations of the form  $w \equiv w'$ ,  $w, w' \in W(X)$  which we treat as a system of formulas of the form  $w \equiv w'$  on  $W(X)$ . Since  $w$  and  $w'$  are formulas in  $W(X)$ , then  $w = w(x_1, \dots, x_n)$ ,  $w' = w'(x_1, \dots, x_n)$ .

**Definition 3.1** A point  $\bar{a} = (a_1, \dots, a_n) \in H^n$  is a solution of  $w \equiv w'$  in the algebra  $H$  if  $w(a_1, \dots, a_n) = w'(a_1, \dots, a_n)$ . A point  $\mu \in \text{Hom}(W(X), H)$  is a solution of  $w \equiv w'$  if  $w(\mu(x_1), \dots, \mu(x_n)) = w'(\mu(x_1), \dots, \mu(x_n))$ .

The equality  $w(\mu(x_1), \dots, \mu(x_n)) = w'(\mu(x_1), \dots, \mu(x_n))$  means that the pair  $(w, w')$  belongs to  $\text{Ker}(\mu)$ . In other words, a point  $\mu$  is a solution of the equation  $w \equiv w'$  if this formula belongs to the kernel of the point  $\mu$ . Thus we say that  $w \equiv w'$  belongs to the kernel of a point if and only if the pair  $(w, w')$  belongs to this kernel. The kernel  $\text{Ker}(\mu)$  is a congruence of the algebra  $W(X)$ , and the quotient algebra  $W(X)/\text{Ker}(\mu)$  is defined.

Let now  $T$  be a system of equations in  $W(X)$  and  $A$  a set of points in  $\text{Hom}(W(X), H)$ . Set the Galois correspondence by

$$T'_H = A = \{\mu : W(X) \rightarrow H \mid T \subset \text{Ker}(\mu)\}$$

$$A'_H = T = \{(w \equiv w') \mid (w, w') \in \bigcap_{\mu \in A} \text{Ker}(\mu)\}.$$

**Definition 3.2** A set  $A$  in the affine space  $\text{Hom}(W(X), H)$  is called an algebraic set if there exists a system of equations  $T$  in  $W(X)$  such that each point  $\mu$  of  $A$  satisfies all equations from  $T$ . A congruence  $T$  in  $W(X)$  is called  $H$ -closed if there exists  $A$  such that  $A'_H = T$ .

We can rewrite the Galois correspondence through the values of formulas:

$$T'_H = A = \bigcap_{(w, w') \in T} \text{Val}_H^X(w \equiv w').$$

$$A'_H = T = \{w \equiv w' \mid A \subset \text{Val}_H^X(w \equiv w')\}.$$

The geometry obtained via this correspondence is an equational geometry grounded on algebra  $H$  in  $\Theta$ . However, there are no reasons to restrict ourselves with equational predicates looking at the images of the formulas in the affine space. We can look at arbitrary first order formulas as at equations, and since arbitrary formulas are the elements of  $\tilde{\Phi} = (\Phi(X), X \in \Gamma)$ , we shall replace in all consideration the free algebra  $W(X)$  by the extended Boolean algebra  $\Phi(X)$ .

The sets of equations are defined as arbitrary subsets in  $\Phi(X)$ , the finite dimensional affine space  $H^n$  is the same as in equational case, and it remains to define the geometric objects, that is the images of the formulas  $u \in \Phi(X)$  in the Galois correspondence. This can be done because, as we know, the equalities  $M_X$ ,  $X \in \Gamma$  represent the free generators of  $\tilde{\Phi}$  and, thus the value homomorphism  $\text{Val}_H^X$  can be extended from equalities to arbitrary formulas  $u \in \Phi(X)$ .

Let  $\mu : W(X) \rightarrow H$  be a point. Along with the classical kernel  $\text{Ker}(\mu)$  we define its logical kernel.

**Definition 3.3** A formula  $u \in \Phi(X)$  belongs to the logical kernel  $\text{LKer}(\mu)$  of a point  $\mu$  if and only if  $\mu \in \text{Val}_H^X(u)$ .

It can be verified that the logical kernel  $LKer(\mu)$  is always a Boolean ultra-filter of  $\Phi(X)$  [32].

Since we consider each formula  $u \in \Phi(X)$  as an "equation" and  $Val_H^X(u)$  as a value of the formula  $u$  in the algebra  $Bool(W(X), H)$ , then  $Val_H^X(u)$  is a set of points  $\mu : W(X) \rightarrow H$  satisfying the "equation"  $u$ . We call  $Val_H^X(u)$  solutions of the equation  $u$ . We also say that the formula  $u$  holds true in the algebra  $H$  at the point  $\mu$ .

*We call the obtained geometry associated to an arbitrary variety  $\Theta$  and  $H \in \Theta$  the logical geometry.*

In order to establish in this case the Galois correspondence we shall replace the kernel  $Ker(\mu)$  by the logical kernel  $LKer(\mu)$ . Let  $T$  be a set of formulas in  $\Phi(X)$  and  $A$  a set of elements in  $Bool(W(X), H)$ . Define

$$T_H^L = A = \{\mu : W(X) \rightarrow H \mid T \subset LKer(\mu)\},$$

$$A_H^L = T = \bigcap_{\mu \in A} LKer(\mu)$$

The same Galois correspondence can be rewritten as

$$T_H^L = A = \bigcap_{u \in T} Val_H^X(u).$$

$$A_H^L = T = \{u \in \Phi(X) \mid A \subset Val_H^X(u)\}.$$

**Definition 3.4** *A set  $A$  in the affine space  $Hom(W(X), H)$  is called an elementary set if there exists a system of formulas  $T$  in  $\Phi(X)$  such that each point  $\mu$  of  $A$  satisfies all formulas from  $T$ . In other words,  $A = A_H^{LL} = T_H^L$  is fulfilled for elementary sets.*

**Definition 3.5** *A set of formulas  $T \subset \Phi(X)$  such that  $T = T_H^{LL} = A_H^L$  is called an  $H$ -closed Boolean filter in  $\Phi(X)$ .*

**Remark 3.6** *The set of formulas  $T$  which defines an elementary set  $A$  can be infinite.*

**Remark 3.7** *Elementary sets in the model theory are usually called definable sets. Since in the geometrical approach they are tightly connected with elementary theories, we use the term "elementary set" instead of "definable set".*

**Remark 3.8** *The formulas from  $T \subset \Phi(X)$  may contain free generators from different  $X_i$ ,  $i \in \Gamma$ . For example, the formula*

$$u = s_*(y_1 \equiv y_2) \vee (x_3 \equiv x_4),$$

*where  $X = \{x_1, x_2, x_3, x_4\}$ ,  $Y = \{y_1, y_2\}$  and  $s(y_1) = x_1$ ,  $s(y_2) = x_2$ , belongs to  $\Phi(X)$ .*

## 4 Model theoretic types

In this section we have to recall, first, the well-known definitions from model theory. In our exposition, we mainly follow the standard model theory course by [28], see also [27], [39], etc. We assume that the precise definition of an  $\mathbb{L}$ -structure is known. Basically, an  $\mathbb{L}$ -structure is a pair  $(\mathbb{L}, M)$ , where  $\mathbb{L}$  is a language and  $M$  is a set, called the domain of the structure. Any language may contain functional symbols, symbols of relations, and special symbols called constants. Given an  $\mathbb{L}$ -structure, all these symbols are interpreted (realized) on the domain  $M$ . So any  $\mathbb{L}$ -structure can be considered as a triple  $(\mathbb{L}, M, f)$ , where  $f$  is an interpretation function.

Formulas of  $\mathbb{L}$  are built inductively from atomic formulas, using the symbols of  $\mathbb{L}$ , symbols of variables  $x_1, x_2, \dots$ , the equality symbol  $\equiv$ , the Boolean connectives  $\wedge, \vee, \neg$ , the quantifiers  $\exists$  and  $\forall$ , and parentheses  $(, )$ . We suppose that the interpretation of symbol  $\equiv$  is always equality on  $M$ .

A variable  $x$  *occurs freely* in a formula  $u$  if it is not bounded by quantifiers  $\exists x$  or  $\forall x$ . A formula  $u$  is called a *sentence* (or a closed formula) if it has no free variables. If  $u(x_1, \dots, x_n)$  is a formula in free variables  $x_1, \dots, x_n$  then its closure  $\bar{u}$  is any sentence produced from  $u$  by bounding all free variables by quantifiers.

Let  $\mathbb{M}$  be an  $\mathbb{L}$ -structure. For an  $\mathbb{L}$ -formula  $u$  one writes  $\mathbb{M} \models u$  to say that the value of  $u$  under the interpretation  $f$  is true. The value ("true" or "false") under interpretation  $f(x_i) = a_i, i = 1, \dots, m, a_i \in M$  of a formula  $u = u(x_1, \dots, x_m)$  is defined inductively, using Tarski schema. Each  $\mathbb{L}$ -sentence is either true or false on the whole  $\mathbb{M}$ . Let  $u(x_1, \dots, x_n)$  be a formula in free variables  $x_1, \dots, x_n$  which means that all occurrences of other variables in this formula are bounded. If  $u(x_1, \dots, x_n)$  is a formula with free variables  $x_1, \dots, x_n$  and  $\bar{a} = (a_1, \dots, a_n) \in M^n$  then we write  $\mathbb{M} \models u(a_1, \dots, a_n)$  for the true formula under interpretation  $f(x_i) = a_i$ . In this case we say that  $u$  is satisfiable on  $\mathbb{M}$ .

**Definition 4.1** A set  $T$  of  $\mathbb{L}$ -sentences is called an  $\mathbb{L}$ -theory.  $\mathbb{M}$  is a model of the theory  $T$  if  $\mathbb{M} \models u$  for all  $u \in T$ . A theory is satisfiable if it has a model.

Suppose that  $\mathbb{M}$  is an  $\mathbb{L}$ -structure and  $A \subseteq M$ . Let  $\mathbb{L}_A$  be the language obtained by adding to  $\mathbb{L}$  constant symbols for each  $a \in A$ . We can naturally view  $\mathbb{M}$  as an  $\mathbb{L}_A$ -structure by interpreting the new symbols in the obvious way. Let  $Th_A(\mathbb{M})$  be the set of all  $\mathbb{L}_A$ -sentences true in  $\mathbb{M}$ , that is the  $\mathbb{L}_A$ -theory of the model  $\mathbb{M}$ .

**Definition 4.2** If  $\mathbb{L}$  is a first order language, then  $Th_A(\mathbb{M})$  is called the elementary theory of  $M$ .

**Definition 4.3** Let  $P = \{u_i(x_1, \dots, x_n)\}$  be a set of  $\mathbb{L}_A$ -formulas in free variables  $x_1, \dots, x_n$ . We call  $P$  an  $n$ -type (partial  $n$ -type) if  $P \cup Th_A(\mathbb{M})$  is satisfiable. We say that  $P$  is a complete  $n$ -type if  $u \in P$  or  $\neg u \in P$  for all  $\mathbb{L}_A$ -formulas  $u$  with free variables from  $x_1, \dots, x_n$ .

So, the data for a type  $P$  is a structure  $\mathbb{M}$  and a subset of constants  $A \subseteq M$ . If  $\mathbb{M}$  is any  $\mathbb{L}$ -structure,  $A \subseteq M$ , and  $\bar{a} = (a_1, \dots, a_n) \in M^n$ , let  $tp^{\mathbb{M}}(\bar{a}/A) = \{u(x_1, \dots, x_n) \in \mathbb{L}_A : \mathbb{M} \models u(a_1, \dots, a_n)\}$ . Then,  $tp^{\mathbb{M}}(\bar{a}/A)$  is a complete  $n$ -type.

**Definition 4.4** We say that a complete  $n$ -type  $P$  is realized in  $\mathbb{M}$  if there is  $\bar{a} = (a_1, \dots, a_n) \in M^n$  such that  $P = tp^{\mathbb{M}}(\bar{a}/A)$ .

Denote the sets of all complete realizable  $n$ -types over  $M$  by  $S_A^n(\mathbb{M})$ . In case  $A = M$  we denote this set by  $S^n(\mathbb{M})$ .

**Problem 4.5** Suppose that for two structures  $\mathbb{M}_1$  and  $\mathbb{M}_2$  the sets of complete realizable  $n$ -types  $S^n(\mathbb{M}_1)$  and  $S^n(\mathbb{M}_2)$  coincide for every  $n$ . What can be said about  $\mathbb{M}_1$  and  $\mathbb{M}_2$ ? How far are these structures from being isomorphic?

**Remark 4.6** Topologically, this question is very close to the following one: suppose two structures have isomorphic Stone spaces (i.e., the spaces of complete realizable  $n$ -types  $S^n(\mathbb{M})$ ) for each  $n$ . What can be said about relations between the structures in this case?

Problem 4.5 is a generalization of the problem about elementary equivalence of structures. Loosely speaking we ask how distant can algebraic structures be if not only their logical descriptions coincide, but coincide also the logical descriptions of particular elements from these structures. This question can be specialized to specific varieties of algebras  $\Theta$  and to specific algebras in  $\Theta$ .

## 5 Algebraization of model theoretic types

Define an algebraization of the notion of type. Let  $X^0$  be an infinite set of variables. Let  $H$  be an algebra from a variety of algebras  $\Theta$ . Let the set of constants equal  $H$ , that is we consider algebras  $G$  from the variety  $\Theta^H$  of  $H$ -algebras. For example, if  $\Theta$  is the variety of commutative and associative rings with the unit and  $K$  is a field, then  $\Theta^K$  is the variety of algebras over the field  $K$ .

In our case, the free algebras in  $\Theta^H$  have the form  $W(X^0) = W'(X^0) * H$ , where  $W'(X^0)$  is the free algebra in  $\Theta$  and  $*$  stands for the free product in  $\Theta$ .

Let  $\Phi(X^0)$  be the one-sorted Halmos algebra of formulas associated with the variety  $\Theta^H$ . Recall that  $\Phi(X^0)$  is constructed in the following way. We consider the signature consisting of symbols of Boolean connectives, existential quantifiers  $\exists x$ ,  $x \in X^0$ , equalities of the form  $w(x_1, \dots, x_n) \equiv w'(x_1, \dots, x_n)$ , where  $w, w'$  belongs to  $W(X)$ ,  $X$  runs all finite subsets of  $X^0$ , and symbols of operations  $s : W(X) \rightarrow W(X)$ , for every  $X$ . Let us take the absolutely free algebra over equalities in this signature. The quotient of this algebra by the Lindenbaum-Tarski congruence is  $\Phi(X^0)$ . The pair  $(\Phi(X^0), H)$  plays the role of  $\mathbb{L}_M$ -structure  $\mathbb{M}$ , where  $M = H$ .

Now we recall the Galois correspondence from the previous section in the case when  $\tilde{\Phi} = (\Phi(X), X \in \Gamma)$  is a one-sorted Halmos algebra  $\Phi(X^0)$ ,  $X^0$  is infinite. Let  $T$  be a set of formulas in  $\Phi(X^0)$ . We have

$$T_H^L = A = \{\mu : W(X) \rightarrow H \mid T \subset LKer(\mu)\},$$

$$A_H^L = T = \bigcap_{\mu \in A} LKer(\mu)$$

In particular,  $u \in T$  if and only if  $A \subset Val_H^X(u)$ .

Let  $X = \{x_1, \dots, x_n\}$  be a finite subset in  $X^0$ . We shall define  $X$ - $MT$ -type ( $MT$ -type for short) of the point  $\mu \in \text{Hom}(W(X), H) \cong H^n$ .

For each point  $\mu : W(X) \rightarrow H$  consider the set of points  $A_\mu$  defined by: a point  $\nu : W(X^0) \rightarrow H$  belongs to  $A_\mu$  if  $\nu(x) = \mu(x)$  for  $x \in X$  and  $\nu(y)$  is an arbitrary element in  $H$ . Define

$$T_\mu = (A_\mu)_H^L = \bigcap_{\nu \in A_\mu} LKer(\nu).$$

In other words  $T_\mu$  is the set of all formulas  $u \in \Phi(X^0)$  which hold on the points from  $A_\mu$ . This means that  $u \in T_\mu$  if  $A_\mu \subset \text{Val}_H^{X^0}(u)$ . Since every logical kernel is an ultrafilter, the set  $T_\mu$  is a filter.

**Definition 5.1** We call the filter  $T_\mu$  an  $MT$ -type of the point  $\mu$ .

**Remark 5.2** Let us compare Definitions 4.1 – 4.4 and Definition 5.1. In the definition 5.1 we consider an  $MT$ -type of the point  $\bar{a} = (a_1, \dots, a_n)$ , where  $\mu(x_i) = a_i$ ,  $a_i \in H$  for  $x_i \in X$ , as the set of all formulas  $u$  which hold true on the point  $\mu$  (i.e., on the point  $\bar{a}$ ). Therefore, the type of a point in our definition is always a filter.

On the other hand, by the definition 4.4 the type of the point  $tp^{\mathbb{H}}(\bar{a}) = tp^{\mathbb{H}}(\mu)$ , where  $\mu(x_i) = a_i$ ,  $i = 1, \dots, n$  is the set of the satisfiable in the point  $\mu$  formulas of the form  $u = u(x_1, \dots, x_n, y_1, \dots, y_k)$ , where only  $x_i$  are free variables. This is a subset of  $T_\mu$  and thus an  $MT$ -type  $T_\mu$  is somewhat bigger than the corresponding  $tp^{\mathbb{H}}(\mu)$ .

**Remark 5.3** The similar situation holds with the definition of the elementary theory of an algebra  $H$ .

We will consider elementary theory of  $H$  as the set of all formulas  $u$  true in every point  $\mu : \text{Hom}(W(X), H)$ .

On the other side, according to the modal-theoretic Definition 4.1 the elementary theory of  $H$  is smaller and consists of closed formulas true in  $H$ . Since every formula  $u$  true in  $H$  is equivalent to its closure  $\bar{u}$ , then by abuse of language we use the same notation  $Th(H)$  for the elementary theory of  $H$  in both cases. So,

$$Th(H) = \bigcap_{\mu} T_\mu,$$

where  $\mu \in \text{Hom}(W(X), H)$ .

This situation is typical for algebraic logic and geometry where the free variables do not play the same role as in logic and model theory.

Denote the system of all  $MT$ -types  $T_\mu$  of the algebra  $H$  by  $S_H^X$ . Here,  $\mu : W(X) \rightarrow H$ , and  $X$  runs all finite subsets of  $X^0$ .

Given finite subset  $X \subset X^0$  and a point  $\mu : W(X) \rightarrow H$ , define  $s = s^\mu : W(X^0) \rightarrow W(X^0)$ , where  $W(X^0) = W'(X^0) * H$ , by letting  $s(x_i) = \mu(x_i)$ , if  $x_i \in X$ , and  $s(y) = y$  for  $y \in Y^0 = X^0 \setminus X$ . Let  $s_*^\mu : \Phi(X^0) \rightarrow \Phi(X^0)$  be the corresponding map of Halmos algebras.

**Proposition 5.4** A formula  $u \in \Phi(X^0)$  belongs to  $T_\mu$  if and only if  $s_*^\mu u$  belongs to the elementary theory  $Th(H)$ .

*Proof.* Let  $s_*^\mu u$  belong to the elementary theory  $Th(H)$ . We shall prove that  $u \in T_\mu$ . Thus, we shall check that  $A_\mu \subset Val_H^{X^0}(u)$ . Let  $\nu \in A_\mu$ . Let  $\delta : W(X^0) \rightarrow H$  be an arbitrary point in  $Hom(W(X^0), H)$ . Then, for  $x_i \in X$ , we have  $\delta s^\mu(x_i) = \delta(\mu(x_i)) = \mu(x_i)$  since  $\delta$  fixes constants. Correspondingly,  $\delta s^\mu(y_i) = \delta(y_i)$ . Thus we can choose  $\delta$  such that  $\delta s^\mu = \nu$  for any  $\nu \in A_\mu$ . Since  $s_*^\mu u \in Th(H)$ , then  $\delta$  lies in  $Val_H^{X^0}(s_*^\mu u) = s_*^\mu Val_H^{X^0}(u)$ . The latter equality means, by definition, that  $\delta s^\mu$  lies in  $Val_H^{X^0}(u)$ . Hence,  $A_\mu \subset Val_H^{X^0}(u)$ .

Conversely, let  $u \in T_\mu$ . We shall prove that  $s_*^\mu u$  belongs to the elementary theory  $Th(H)$ . So we have to check that any point  $\delta$  satisfies  $s_*^\mu u$ . Consider  $\delta s^\mu$ . This point belongs to  $A_\mu$ . Hence  $\delta s^\mu$  lies in  $Val_H^{X^0}(u)$ . This means that  $\delta$  lies in  $s_*^\mu Val_H^{X^0}(u) = Val_H^{X^0}(s_*^\mu u)$ . Thus, an arbitrary point  $\delta$  belongs to  $Val_H^{X^0}(s_*^\mu u)$  and  $s_*^\mu u$  lies in  $Th(H)$ .  $\square$

Let  $u = u(x_1, \dots, x_n, y_1, \dots, y_k)$  be a formula in  $\Phi(X^0)$  such that  $x_i \in X$ ,  $y_i \in Y$ , and all occurrences of  $x_i$  are free, all occurrences of  $y_i$  are bounded. We call such a formula special.

Let  $u$  be a special formula. It can be seen that  $s_*^\mu u$  replaces all occurrences of free variables  $x_i$  by the their images  $h_i \in H$  under the homomorphism  $s^\mu$ . Hence  $s_*^\mu u$  has all variables bounded, i.e.,  $s_*^\mu u$  is a sentence.

Any  $MT$ -type is complete with respect to special formulas. Indeed, let  $u$  be a special formula and let  $u \notin T_\mu$ . Consider  $\neg u$ . We have  $s_*^\mu(\neg u) = \neg s_*^\mu(u)$ . By Proposition 5.4,  $s_*^\mu(u)$  does not hold in  $H$ . Since  $s_*^\mu u$  is a sentence, the formula  $\neg s_*^\mu(u)$  holds in  $H$ . Hence,  $s_*^\mu(\neg u)$  holds in  $H$  and thus belongs to  $Th(H)$ . Then  $\neg u \in T_\mu$  according to Proposition 5.4.

Suppose now that for two algebras  $H_1$  and  $H_2$  the sets  $S_{H_1}^X$  and  $S_{H_2}^X$  of  $MT$ -types  $T_\mu$  coincide. Every  $MT$ -type contains the corresponding model theoretic  $n$ -type, where  $n = |X|$ . So the problem 4.5 can be restated as what can be said about the closeness of algebras  $H_1$  and  $H_2$  if  $T_\mu$  and  $T_\nu$  coincide?

From now on, one can build the type theory from the positions of one-sorted algebraic logic. In the next section we consider a more geometric approach, related to multi-sorted logic and multi-sorted Halmos algebras.

## 6 Logically-geometric types

Let us take the free multi-sorted Halmos algebra of formulas  $\tilde{\Phi} = (\Phi(X), X \in \Gamma)$ , where all  $X$  are finite. Recall the necessary facts from the previous sections.

There is the value homomorphism of multi-sorted Halmos algebras  $Val_H : \tilde{\Phi} \rightarrow Hal_\Theta(H)$ , which induces homomorphisms of extended Boolean algebras  $Val_H^X : \Phi(X) \rightarrow Bool(W(X), H)$ , where  $Hal_\Theta(H) = (Bool(W(X), H), X \in \Gamma)$ . We can write  $Val_H = (Val_H^X, X \in \Gamma)$ . For every  $X$ , the homomorphism  $Val_H^X$  gives rise to a major Galois correspondence of logical geometry between  $H$ -closed congruences in  $\Phi(X)$  and elementary sets in finite dimensional affine spaces  $Hom(W(X), H)$  :

$$T_H^L = A = \{\mu : W(X) \rightarrow H \mid T \subset LKer(\mu)\},$$

$$A_H^L = T = \bigcap_{\mu \in A} LKer(\mu).$$

Let  $Th(H) = (Th^X(H), X \in \Gamma)$  be the multi-sorted representation of the elementary theory of  $H$ . We call its component  $Th^X(H)$  the  $X$ -theory of the



algebra  $H$ . We have:

$$\begin{aligned} \text{Ker}(\text{Val}_H) &= \text{Th}(H), \\ \text{Ker}(\text{Val}_H^X) &= \text{Th}^X(H). \end{aligned}$$

The key diagram which relates logic of different sorts in multi-sorted case is as follows:

$$\begin{array}{ccc} \Phi(X) & \xrightarrow{s_*} & \Phi(Y) \\ \text{Val}_H^X \downarrow & & \downarrow \text{Val}_H^Y \\ \text{Bool}(W(X), H) & \xrightarrow{s_*} & \text{Bool}(W(Y), H) \end{array}$$

Here the upper arrow represent the syntactical transitions in the category  $\text{Hal}_\Theta$ , the lower level does the same with the respect to semantics in  $\text{Hal}_\Theta$ , and the correlation is provided by the value homomorphism.

Recall that a formula  $u \in \Phi(X)$  belongs to the logical kernel  $\text{LKer}(\mu)$  of a point  $\mu$  if and only if  $\mu \in \text{Val}_H^X(u)$ , that is  $u$  lies in  $\text{LKer}(\mu)$  if a point  $\mu$  satisfies the "equation"  $u$ . This is the Boolean ultrafilter, which contains  $\text{Th}^X(H)$ . Indeed, if  $u \in \text{Th}^X(H)$  then  $\text{Val}_H^X(u) = \text{Hom}(W(X), H)$ . In particular,  $\mu \in \text{Val}_H^X(u)$  and  $u \in \text{LKer}(\mu)$ . Thus  $\text{Th}^X(H) \subset \text{LKer}(\mu)$ . Moreover,

$$\text{Th}^X(H) = \bigcap_{\mu} \text{LKer}(\mu).$$

Define now the concept of an  $LG$ -type.

**Definition 6.1** *Every ultrafilter  $T$  in the algebra  $\Phi(X)$  containing  $\text{Th}^X(H)$  is called  $X$ -LG-type.*

**Definition 6.2** *A type  $T$  is called  $X$ -LG-type of the algebra  $H$ , if there is a point  $\mu : W(X) \rightarrow H$  such that  $T = \text{LKer}(\mu)$ .*

In the latter case we also say that the type  $T$  is realized in  $H$ . Since the elementary  $X$ -theory is contained in each  $\text{LKer}(\mu)$  then the elementary  $X$ -theory  $\text{Th}^X(H)$  is contained in each  $X$ -LG-type of  $H$ . Denote the system of all  $X$ -LG-types of the algebra  $H$  by  $S^X(H)$ .

Now we want to explore the geometrical nature of the Galois correspondence. In algebraic geometry, the category of all algebraic sets is an important invariant of the algebra  $H$ . In most cases, this category is dual to the category of coordinated algebras. We want to use similar ideas in the case of logical geometry. The logical kernels take the role played by the radical ideals in classical geometry and the roles of closed congruences in the universal one. So, the types of the points represented by the logical kernels may have similar impact to logical geometry and may be involved in the similar algebraically-geometric ideas.

Two algebras  $H_1$  and  $H_2$  are called *geometrically equivalent* ( $AG$ -equivalent for short) (see [32], [33]) if for every finite  $X$  and  $T$  in  $W(X)$  we have

$$T''_{H_1} = T''_{H_2}.$$

**Definition 6.3 ([38])** Algebras  $H_1$  and  $H_2$  are called *logically equivalent* (*LG-equivalent for short*) if for every finite  $X$  and  $T$  in  $\Phi(X)$  we have

$$T_{H_1}^{LL} = T_{H_2}^{LL}.$$

It can be seen (see [38]), that if two algebras  $H_1$  and  $H_2$  are logically equivalent then they are *elementary equivalent* (i.e.,  $Th(H_1) = Th(H_2)$ ). The converse statement is not true.

**Definition 6.4 ([38])** Algebras  $H_1$  and  $H_2$  in  $\Theta$  are called *LG-isotyped*, if for any finite  $X$ , every  $X$ -LG-type of the algebra  $H_1$  is an  $X$ -LG-type of the algebra  $H_2$  and vice versa.

Thus, the algebras  $H_1$  and  $H_2$  are LG-isotyped if  $S^X(H_1) = S^X(H_2)$  for every  $X \in \Gamma$ . This coincidence clearly implies that they are elementary equivalent.

So, we have the geometric notion of logical equivalence of algebras which generalizes geometric equivalence, and the model theoretic notion of LG-isotypeness. Both of them imply elementary equivalence. The following theorem shows that these two notions coincide.

**Theorem 6.5 ([32])** Algebras  $H_1$  and  $H_2$  are LG-equivalent if and only if they are LG-isotyped.

One can define the category of algebraic sets  $K_\Theta(H)$  and the category of elementary sets  $LK_\Theta(H)$ . The objects of  $K_\Theta(H)$  are of the form  $(X, A)$ , where  $A$  is an algebraic set in  $Hom(W(X), H)$ . If we take for  $A$  the elementary sets, then we are getting to the category of elementary sets  $LK_\Theta(H)$ . The morphisms are of the form

$$[s] : (X, A) \rightarrow (Y, B).$$

Here  $s : W(Y) \rightarrow W(X)$  is a morphism in the category  $\Theta^0$ . The corresponding  $\tilde{s} : Hom(W(X), H) \rightarrow Hom(W(Y), H)$  should be coordinated with  $A$  and  $B$  by the condition: if  $\nu \in A \subset Hom(W(X), H)$ , then  $\tilde{s}(\nu) \in B \subset Hom(W(Y), H)$ . Then the induced mapping  $[s] : A \rightarrow B$  we consider as a morphism  $(X, A) \rightarrow (Y, B)$ .

The category  $K_\Theta(H)$  is a full subcategory in  $LK_\Theta(H)$ . It is known that if two algebras  $H_1$  and  $H_2$  are geometrically equivalent, then the categories of algebraic sets  $K_\Theta(H_1)$  and  $K_\Theta(H_2)$  are isomorphic. A similar fact is valid with respect to categories of elementary sets. Namely,

**Theorem 6.6 ([38])** If the algebras  $H_1$  and  $H_2$  are LG-isotyped then the categories  $LK_\Theta(H_1)$  and  $LK_\Theta(H_2)$  are isomorphic.

## 7 Problems

In Sections 5 and 6 we described *MT*-types and *LG*-types. Now we want to compare these notions.

Recall that *MT*-types are defined for points  $\mu : W(X) \rightarrow H$  of the affine space  $Hom(W(X), H)$ . However, the formulas from any *MT*-type  $T_\mu$  lie in the algebra of formulas  $\Phi(X^0)$ , where  $X^0$  is an infinite set. It is important to note,

that the algebra  $H$  from the given variety of algebras  $\Theta$  is treated as the algebra of constants.

In the case of  $LG$ -types, we consider finite sets  $X$  in  $X^0$  and the multi-sorted algebra of formulas  $\tilde{\Phi} = (\Phi(X), X \in \Gamma)$ , where all  $X$  are finite. The  $X$ - $LG$ -type of the point  $\mu : W(X) \rightarrow H$  is  $LKer(\mu)$ , which is calculated in the algebra  $\Phi(X)$ . This is one of the differences in two approaches. We shall also remember that the formulas from  $T \subset \Phi(X)$  may contain free generators from different  $X$ , where  $X \in \Gamma$  (see Remark 3.8).

**Problem 7.1** *Compare MT-isotypeness and LG-isotypeness. In other words, are there algebras  $H_1$  and  $H_2$  such that they are MT-isotyped but not LG-isotyped, or such that they are LG-isotyped but not MT-isotyped?*

Problems 7.2 and 7.3 are devoted to  $LG$ -types.

**Problem 7.2** *Let  $F_n$  be a free group of the rank  $n > 1$  and  $H$  be a finitely generated group. Is it true that if  $F_n$  and  $H$  are LG-isotyped then they are isomorphic?*

**Problem 7.3** *Are there LG-isotyped groups  $H_1$  and  $H_2$  such that  $H_1$  is finitely generated and  $H_2$  is an arbitrary non finitely generated group?*

C. Perin and R. Sklinos [30] proved that if for a non-abelian free group there is the equality  $T_\mu = T_\nu$  then  $\mu = \sigma\nu$  for some automorphism  $\sigma$  of  $H$ .

**Problem 7.4** *What are the varieties  $\Theta$  such that for arbitrary free algebra  $H = W(X)$  from  $\Theta$  the equality  $T_\mu = T_\nu$  implies  $\mu = \sigma\nu$ ?*

Similar question for  $LG$ -types and free groups is of great interest.

**Problem 7.5** *Is it true that for a given free non-abelian group the equality  $LKer(\mu) = LKer(\nu)$  implies  $\mu = \sigma\nu$ ?*

Problem 7.5 has positive solution for the case of free abelian groups (G. Zhitomirski, unpublished).

Note that the group of automorphisms of an algebra  $H$  acts on the affine space  $Hom(W(X), H)$ , and each elementary set is invariant under this action. If for the algebra  $H$  there are only a finite number of  $Aut(H)$ -orbits in  $Hom(W(X), H)$  for every  $X$ , then there are only finite number of realizable  $LG$ -types in  $\Phi(X)$ . It can be shown that for free abelian groups of the exponent  $p$  this property is satisfied. It would be interesting to look for non-abelian examples.

**Problem 7.6** *Find examples of algebras  $H$  such that for every  $X$  there are only a finite number of  $Aut(H)$ -orbits in  $Hom(W(X), H)$ .*

## References

- [1] M. Amer, T.S. Ahmed, *Polyadic and cylindric algebras of sentences*, (English summary) MLQ Math. Log. Q. **52**, (2006), no. 5, p. 444-449.

- [2] H. Andreka, I. Nemeti, I. Sain, *Algebraic logic. Handbook of philosophical logic*, Kluwer Acad. Publ., Dordrecht, **2**, (2001), p. 133-247.
- [3] G.Baumslag, A. Myasnikov, V. Remeslennikov, *Algebraic geometry over groups I*, J. of Algebra, **219:1**, (1999), 16–79.
- [4] C.C. Chang, H.J. Keisler, *Model Theory*, North-Holland Publ. Co., (1973).
- [5] E. Daniyarova, I. Kazachkov, V. Remeslennikov, *Algebraic geometry over a free metabelian Lie algebra. I. U-algebras and universal classes*. (Russian) Fundam. Prikl. Mat. **9**, (2003), no. 3, p. 37–63; translation in J. Math. Sci. (N. Y.) **135** (2006), no. 5.
- [6] E. Daniyarova, I. Kazachkov, V. Remeslennikov, *Algebraic geometry over a free metabelian Lie algebra. II. The finite field case*. (Russian) Fundam. Prikl. Mat. **9**, (2003), no. 3, p. 65–87; translation in J. Math. Sci. (N. Y.) **135**, (2006), no. 5.
- [7] E. Daniyarova, A. Myasnikov, V. Remeslennikov, *Unification theorems in algebraic geometry. Aspects of infinite groups*, Algebra Discrete Math., **1**, World Sci. Publ., Hackensack, NJ, (2008), 80–111.
- [8] E.Daniyarova, A.Miasnikov, V.Remeslennikov, *Algebraic geometry over algebraic structures II: Foundations*, J. Algebra, submitted, arXiv:01002.3562v1 [math.AG]
- [9] E.Daniyarova, A.Miasnikov, V.Remeslennikov, *Algebraic geometry over algebraic structures III: Equationally Noetherian property and compactness*, arXiv: 1002.4243v1 [math.AG]
- [10] P.R. Halmos, *Algebraic logic*, New York, (1969). Papers [11]–[14] are reprinted in [10].
- [11] P.R. Halmos, *Algebraic logic. I. Monadic Boolean algebras*. Compositio Math. **12**, (1956), p. 217–249.
- [12] P.R. Halmos, *Algebraic logic. II. Homogeneous locally finite polyadic Boolean algebras of infinite degree*. Fund. Math. **43**, (1956), p. 255–325.
- [13] P.R. Halmos, *Algebraic logic. III. Predicates, terms, and operations in polyadic algebras*. Trans. Amer. Math. Soc. **83**, (1956), p. 430–470.
- [14] P.R. Halmos, *Algebraic logic. IV. Equality in polyadic algebras*. Trans. Amer. Math. Soc. **86**, (1957), 1–27.
- [15] L. Henkin, J.D. Monk, A. Tarski, *Cylindric Algebras*, North-Holland Publ. Co. (1971, 1985).
- [16] P.J. Higgins, *Algebras with a scheme of operators*, Math. Nachr. **27**, (1963), p. 115–132.
- [17] W. Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications **42**, Cambridge University Press, Cambridge, (1993).

- [18] J.M. Font, R. Jansana, D. Pigozzi, *A survey of abstract algebraic logic. Abstract algebraic logic*, Part II (Barcelona, 1997). *Studia Logica* **74**, (2003), no. 1-2, p. 13–97.
- [19] J.M. Font, R. Jansana, D. Pigozzi, *Update to "A survey of abstract algebraic logic"*, *Studia Logica* **91**, (2009), no. 1, p. 125–130,
- [20] O. Kharlampovich, A. Myasnikov, *Irreducible affine varieties over free groups I: Irreducibility of quadratic equations and Nullstellensatz*, *J. of Algebra*, **200:2**, (1998), 472–516.
- [21] O. Kharlampovich, A. Myasnikov, *Irreducible affine varieties over free groups II: Systems in triangular quasi-quadratic form and description of residually free groups*, *J. of Algebra*, **200:2**, (1998), 517–570.
- [22] A. Kvaschuk, A. Myasnikov, V. Remeslennikov, *Algebraic geometry over groups. III. Elements of model theory*. *J. Algebra*, **288:1**, (2005), p. 78–98.
- [23] A.G. Kurosh, *Lectures in general algebra*. International Series of Monographs in Pure and Applied Mathematics, Vol. 70 Pergamon Press, Oxford-Edinburgh-New York, 1965.
- [24] L. LeBlanc, *Nonhomogeneous polyadic algebras*, *Proc. Amer. Math. Soc.* **13**, (1962), p. 59–65.
- [25] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag, (1971).
- [26] A.I. Malcev *Algebraic Systems*, Springer-Verlag, (1973).
- [27] Yu.I. Manin, *A course in mathematical logic for mathematicians*. Second edition. Chapters I–VIII translated from the Russian by Neal Koblitz. With new chapters by Boris Zilber and the author. Graduate Texts in Mathematics, 53. Springer, New York, 2010. xviii+384 pp.
- [28] D. Marker, *Model Theory: An Introduction*, Springer Verlag, 2002, 360pp.
- [29] A. Myasnikov, V. Remeslennikov, *Algebraic geometry over groups II, Logical foundations*, *J. of Algebra*, **234:1**, (2000), 225–276.
- [30] C. Perrin, R. Sklinos, *Homogeneity in the free group*, arXiv: 1003.4095v1 [math GM], (2010).
- [31] B. Plotkin, *Algebraic geometry in First Order Logic*, *Sovremennaja Matematika and Applications* **22** (2004), p. 16–62. *Journal of Math. Sciences*, **137**, n.5, (2006), p. 5049– 5097. [http:// arxiv.org/ abs/ math GM/0312485](http://arxiv.org/abs/math.GM/0312485).
- [32] B. Plotkin, *Isotyped algebras*. Arxiv: math.LO/0812.3298v2 (2009). Submitted.
- [33] B. Plotkin, *Seven lectures on the universal algebraic geometry*, Preprint,(2002), Arxiv:math, GM/0204245, 87pp.
- [34] B. Plotkin, *Some results and problems related to universal algebraic geometry*, *International Journal of Algebra and Computation*, **17(5/6)**, (2007), p. 1133–1164.

- [35] B. Plotkin, *Universal algebra, algebraic logic and databases*. Kluwer Acad. Publ., 1994.
- [36] B. Plotkin, G. Zhitomirski, *Automorphisms of categories of free algebras of some varieties*, J. Algebra, **306**, (2006), no. 2, p. 344–367.
- [37] B. Plotkin, G. Zhitomirski, *On automorphisms of categories of universal algebras*, Internat. J. Algebra Comput. **17**, (2007), no. 5-6, p. 1115–1132.
- [38] B. Plotkin, G. Zhitomirski, *Some logical invariants of algebras and logical relations between algebras*, Algebra and Analysis, **19:5**, (2007), p. 214–245, St. Petersburg Math. J., **19:5**, (2008), p. 859–879.
- [39] B. Poizat, *A course in model theory. An introduction to contemporary mathematical logic*. Translated from the French by Moses Klein and revised by the author. Universitext. Springer-Verlag, New York, (2000), xxxii+443 pp.
- [40] E. Rips, Z. Sela, *Cyclic splittings of the finitely presented groups and the canonical JSJ decomposition*, Ann. of Math., **146:1**, (1997), p. 53–109.
- [41] Z. Sela, *Diophantine geometry over groups I*, IHES, **93**, (2001), p. 31–105.
- [42] I.R. Shafarevich, *Basic algebraic geometry*, Berlin, Springer-Verlag, (1974).